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Rapport ZW 1954-012.

Voordracht door Prof.K.Yano in de serie Actualiteiten op 30 October 1954.

On pseudo-kählerian spaces of constant holomorphic curvature.

1. Kählerian spaces.

A kählerian space is defined as a complex space in which a Hermitian metric

(1) $$ds^2 = 2\, g_{\bar\lambda\kappa}\, dz^{\bar\lambda} dz^{\kappa} \qquad (g_{\bar\lambda\kappa} = g_{\kappa\bar\lambda} = \overline{g_{\bar\kappa\lambda}}).$$

satisfying

(2) $$\partial_{\bar\mu} g_{\bar\lambda\kappa} = \partial_{\bar\lambda} g_{\bar\mu\kappa} ,$$

is given, where

$$z^{\bar\lambda} = \overline{z^{\lambda}}, \qquad \partial_\mu = \frac{\partial}{\partial z^\mu}, \qquad \partial_{\bar\mu} = \frac{\partial}{\partial \bar z^\mu}$$

and the indices $\kappa, \lambda, \mu, \ldots$ run over the range $1, 2, \ldots, n$ and the indices $\bar\kappa, \bar\lambda, \bar\mu, \ldots$ the range $\bar 1, \bar 2, \ldots, \bar n$.

If we put

(3) $$g_{ih} = \begin{pmatrix} 0 & g_{\bar\lambda\kappa} \\ g_{\lambda\bar\kappa} & 0 \end{pmatrix}, \qquad F_i^{\cdot h} = \begin{pmatrix} i\delta_\lambda^\kappa & 0 \\ 0 & -i\delta_{\bar\lambda}^{\bar\kappa} \end{pmatrix},$$

we can easily verify that g_{ih} and $F_i^{\cdot h}$ are related by

(4) $$g_{lk} F_i^{\cdot l} F_h^{\cdot k} = g_{ih}$$

and the tensor $F_i^{\cdot h}$ satisfies

(5) $$F_j^{\cdot h} F_i^{\cdot j} = -A_i^h ,$$

where the indices h, i, j, k, l, m run over the range $1, 2, \ldots, n, \bar 1, \ldots, \bar n$. Equations (4) and (5) show that the tensor

(6) $$F_{ih} = F_i^{\cdot l} g_{lh}$$

is anti-symmetric in its two lower indices. The equations (4), (5) and (6) are invariant and remain valid if the coordinate system (h) is not taken special as defined by (3) but general, h, i, j = 1,...,2n.

Moreover, the condition (2) is expressed as

(7)
$$\nabla_j F_i^{\cdot k} = 0 \quad \text{or} \quad \nabla_j F_{ik} = 0$$

where ∇_j denotes the covariant differentiation with respect to the Christoffel symbols $\left\{ {k \atop j\,i} \right\}$ formed with g_{ik}.

2. Pseudo-kählerian spaces.

Now consider a 2n-dimensional real space in which two tensors g_{ik} and $F_i^{\cdot k}$ satisfying (4), (5) and (7) are given. We call such a space a pseudo-kählerian space.

If a pseudo-kählerian space is of class C^{ω}, then we can prove that a pseudo-kählerian space is a kählerian space, but if a pseudo-kählerian space is only of class C^r (r = 0, 1, 2,..., ∞) the proof is not yet given. So we call it a pseudo-kählerian space for the time being.

3. Constant sectional curvature.

In a riemannian space, consider two linearly independent vectors u^κ and v^κ at a fixed point P. These vectors determine a 2-dimensional plane-element. Consider all the geodesics which pass through the point P and are tangent to the 2-dimensional plane-element determined by u^κ and v^κ. Then these geodesics constitute a 2-dimensional subspace which passes through the point P and are tangent to the 2-dimensional plane-element determined by u^κ and v^κ. The Gaussian curvature or total curvature of this 2-dimensional subspace is called the sectional curvature of the Riemannian space at the point P with respect to the section determined by two linearly independent vectors u^κ and v^κ.

If we denote by $K_{\nu\mu\lambda\kappa}$ the curvature tensor of the riemannian space and assume that the two vectors u^κ and v^κ are unit vectors and mutually orthogonal, then this sectional curvature K is given by the formula

(8)
$$K = - K_{\nu\mu\lambda\kappa}\, u^\nu v^\mu u^\lambda v^\kappa$$

If the sectional curvature is independtdent of the choice of the section at each point, the space is said to be of constant sectional curvature or simply of constant curvature. For a riemannian space of constant curvature, we have the classical

Theorem 1. A riemannian space of constant curvature has the curvature tensor of the form

(9)

$$K_{\nu\mu\lambda\kappa} = \kappa(g_{\nu\kappa} g_{\mu\lambda} - g_{\nu\lambda} g_{\mu\kappa}),$$

where κ is an absolute constant called the curvature of the space.

Now consider a pseudo-kählerian space. If we have a vector v^h at a point P, we can associate to this vector a vector $F_i^{\cdot h} v^i$. Because of the relations (4) and (5), the vector $F_i^{\cdot h} v^i$ has the same length as the vector v^h and is orthogonal to v^h. We call a holomorphic section a section determined by a vector v^h and $F_i^{\cdot h} v^i$ associated to v^h in this way, and call holomorphic sectional curvature the sectional curvature determined by a holomorphic section.

If the holomorphic sectional curvature is independent of the holomorphic choice of the section at each point, the space is said to be of constant holomorphic sectional curvature, or simply of constant holomorphic curvature. For a space of constant holomorphic curvature, we can prove

Theorem 1'. A pseudo-kählerian space of constant holomorphic curvature has the curvature tensor of the form

(10)

$$K_{kjih} = \frac{k}{4}\left[(g_{kh} g_{ji} - g_{ki} g_{jh}) + (F_{kh} F_{ji} - F_{ki} F_{jh}) - 2 F_{kj} F_{ih}\right],$$

where k is an absolute constant.

Theorem 1''. In a pseudo-kählerian space of constant holomorphic curvature, a general sectional curvature κ determined by two mutually orthogonal unit vectors u^h and v^h is given by

(11)

$$\kappa = \frac{k}{4}(1 + 3a^2),$$

where

(12)

$$a = F_{ih} u^i v^h$$

is the cosine of the angle between two unit vectors $F_i^{\cdot h} u^i$ and v^h. Consequently $a^2 \leq 1$. Thus

(13)

$$\frac{k}{4} \leq \kappa \leq k \quad \text{for } k > 0 \quad \text{and} \quad k \leq \kappa \leq \frac{k}{4} \quad \text{for } k < 0.$$

4. Axiom of planes.

In a riemannian space, a two-dimensional subspace is called a plane if a geodesic of the enveloping space joining any two points on the subspace is contained always in the subspace.

If there exists always a plane which passes through a given point and is tangent to a given 2-dimensional section, we say that the riemannian space satisfies the axiom of planes. We have

Theorem 2. A necessary and sufficient condition that a riemannian space satisfies the axiom of planes, is that the space be of constant curvature.

Now take a pseudo-kählerian space. If there exists always a plane which passes through a given point and is tangent to a 2-dimensional holomorphic section, we say that the pseudo-kählerian space satisfies the axiom of holomorphic planes. We can prove

Theorem 2'. A necessary and sufficient condition that a pseudo-kählerian space satisfies the axiom of holomorphic planes is that the space be of constant holomorphic curvature.

5. Free mobility.

If in a riemannian space there exists always a motion which transforms an arbitrarily given point P into another arbitrarily given point $'P$ and an arbitrarily given direction v^h at P into another arbitrarily given direction $'v^h$, then we say that the space admits free mobility. We have

Theorem 3. In order that a riemannian space admits free mobility, it is necessary and sufficient that it be of constant curvature. If in a pseudo-kählerian space there exists always a motion which transforms any two vectors u^h and $F_i^{\cdot h} u^i$ at a point P into any two vectors $'u^h$ and $'F_i^{\cdot h}\, 'u^i$ at any point $'P$, then we say that the space admits holomorphic free mobility. We can prove

Theorem 3'. In order that a pseudo-kählerian space admits holomorphic free mobility, it is necessary and sufficient that it be of constant holomorphic curvature.

6. Conjugate points.

In a riemannian space, consider a geodesic and two points P and Q on it. If all geodesic arcs joining P and a point between P and Q realize the relative minimum but the geodesic arc joining P and Q or P and a point beyond Q do not realize the relative minimum, we say that two points P and Q are consecutive conjugate points.

We have

Theorem 4.

In a riemannian space of constant positive curvature K, the distance between two consecutive conjugate points on a geodesic is constant and is equal to $\pi/\sqrt{\kappa}$.

Corresponding to this, we have

Theorem 4'. In a pseudo-kählerian space of constant positive holomorphic curvature k, the distance between two consecutive conjugate points on a geodesic is constant and is equal to $2\pi/\sqrt{k}$